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GRAVITATIONAL EVOLUTION OF THE LARGE-SCALE PROBABILITY DENSITY DISTRIBUTION: THE EDGEWORTH & GAMMA EXPANSIONS

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ABSTRACT

The gravitational evolution of the cosmic one-point probability distribution function (PDF) has been estimated using an analytic approximation that combines gravitational perturbation theory with the Edgeworth expansion around a Gaussian PDF. Despite the remarkable success of the Edgeworth expansion in modeling the weakly non-linear growth of fluctuations around the peak of the cosmic PDF, it fails to reproduce the expected behaviour in the tails of the distribution. Besides, this expansion is ill-defined as it predicts negative densities and negative probabilities for the cosmic fields. This is a natural consequence of using an expansion around the Gaussian distribution, which is not rigorously well-defined when describing a positive variate, such as the density field. Here we present an alternative to the Edgeworth series based on an expansion around the Gamma PDF. The Gamma expansion is designed to converge when the PDF exhibits exponential tails, which are predicted by Perturbation Theory, in the weakly non-linear regime, and are found in numerical simulations from Gaussian initial conditions. The proposed expansion is better suited for describing a real PDF as it always yields positive densities and the PDF is effectively positive-definite. We compare the performance of the Edgeworth and the Gamma expansions for a wide dynamical range making use of cosmological N-body simulations and assess their range of validity. In general, the Gamma expansion provides an interesting and simple alternative to the Edgeworth series and it should be useful for modeling non-gaussian PDFs in other contexts, such as in the cosmic microwave background.

1. INTRODUCTION

We aim at studying one dimensional probability density functions (PDF), $p(\delta)$, which characterize the statistical properties of a stochastic field at a single point $\delta = \delta(\boldsymbol{r})$. Here we shall concentrate on fluctuations of the density field: $\delta \equiv \rho/\bar{\rho} - 1$ (being $\bar{\rho}$ the mean value of the density field, ρ), smoothed over some fixed scale R. However many of the arguments presented here are quite generic and could also be applied to other contexts. The approach we will follow is to try to recover the full shape of the PDF from the knowledge of its first order moments. This has become by now a textbook problem and there are several standard ways to address it (for a review see Kendall, Stuart & Ord 1994). The solution is not unique, unless there is a well defined hierarchy in the moments and we can define some perturbative approach to the problem.

There are a number of studies to predict the evolution of clustering of density fluctuations, and in particular of its PDF. There have been attempts to derive the PDF from analytic approximations, such as the Zeldovich Approximation (Kofman et al. 1994). Although the Zeldovich Approximation reproduces important aspects of the non-linear dynamics, it is a poor approximation for the PDF and its moments. One way to improve that is to take advantage of the exact non-linear perturbation theory (PT) to estimate the moments (Bernardeau 1992) and use them to derive the PDF from the Edgeworth expansion (Juszkiewicz et al 1995, Bernardeau & Koffman 1995). In this case the PDF is predicted to an accuracy given by the order of the cumulants involved. The Edgeworth expansion has since been used as a tool to characterize the PDF of matter (eg Kim & Strauss 1998, Blinnikov & Moessnew 1998, Pen 1998) and CMB fluctuations (Amendola 1996, Popa 1998). Earlier phenomenological approaches to the construction of the cosmic PDF were developed by Saslaw & Hamilton (1984) and Coles & Jones (1991).

One serious shortcoming of the Edgeworth approach is that the series yields a PDF that is ill-defined. It has negative probability values and assignes non-zero probability to negative densities ($\delta < -1$). This latter problem originates on the fact that the Gaussian PDF, which is the basis for the Edgeworth series, only makes physical sense when the rms fluctuation σ is very small. Here we will try to address some of these problems by exploring the possibility of carrying out expansions around better behaved PDFs, more suitable to yield positive densities when the variance is not that small. We shall concentrate on the Gamma PDF, but other distributions may be handled within the same framework, as our general analysis will show.

Whenever the moment generating function of the PDF is known, it is then possible to reconstruct the full PDF. This has been done previously by using the Legendre transform (Fry 1985) or the inverse Laplace transform (Balian & Schaeffer 1989, Bernardeau 1992, Bernardeau & Kofman 1995). Since for gravitational clustering in the weakly non-linear regime the variance of the PDF is small, one can expand the above transforms to recover the PT limit. In the case of Gaussian initial fluctuations this perturbative expansion yields to the well-known Edgeworth series.

In §2 we shall introduce the Edgeworth series as a generic saddle point approximation to the PDF, along the lines of Fry (1985). An alternative expansion, around the Gamma distribution, is introduced in §3. It is expressed as an orthogonal polynomial expansion of an arbitrary PDF in terms of Laguerre polinomials. The latter are the counterparts to the Hermite polinomials when one expands around an exponential tail instead of a Gaussian one. A detailed comparative analysis of the Edgeworth and Gamma expansions with respect to N-body simulations is presented in §4 and §5. A final discussion with our conclusions is given in §6.

2. EXPANSIONS AROUND A GIVEN PDF

2.1. *One-Point Statistics*

As usual, we shall denote statistical averaging by brackets: $\langle ... \rangle$, so that the expectation value for the moments are:

$$m_J \equiv \langle \delta^J \rangle = \int p(\delta) \delta^J d\delta \tag{1}$$

with J an integer that labels the order of the corresponding moment. $J = 1$ corresponds to the mean, which for the density fluctuation is zero, $m_1 = \langle \delta \rangle = 0$. In this notation the variance, Var (δ), and *rms* fluctuation σ, are defined as: Var $(\delta) \equiv \sigma^2 \equiv m_2 - m_1^2$. It is useful to introduce the *cumulants* k_J:

$$k_J \equiv \langle \delta^J \rangle_c = \left. \frac{d^J \psi(t)}{dt^J} \right|_{t=0} = \left. \frac{d^J \log \mathcal{M}(t)}{dt^J} \right|_{t=0}, \tag{2}$$

where $\psi(t) \equiv \log \mathcal{M}(t)$ is given in terms of the moments of the PDF through $\mathcal{M}(t)$:

$$\mathcal{M}(t) \equiv \langle e^{t\delta} \rangle = \int_{-\infty}^{\infty} p(\delta)\, e^{t\delta}\, d\delta = \sum_J \frac{t^J}{J!} m_J \tag{3}$$

Gravitational clustering from Gaussian initial conditions predicts $< \delta^J >_c \propto < \delta^2 >_c^{J-1}$ on large-scales, thus it is more convenient to introduce the following ratios,

$$S_J \equiv \frac{k_J}{k_2^{J-1}} = \frac{\langle \delta^J \rangle_c}{\langle \delta^2 \rangle_c^{J-1}}. \tag{4}$$

where the *Skewness*, S_3, is the third-order ratio, and the *Kurtosis*, S_4, is the fourth-order one. From $\psi(t)$ we can get back the PDF, $p(\delta)$, by using the inversion formula:

$$p(\delta) = \int_{-i\infty}^{+i\infty} \frac{dt}{2\pi} e^{t\delta + \psi(t)}. \tag{5}$$

Consider two differentiable PDFs, $p_1(\delta)$ and $p_2(\delta)$, with cumulants $k_J^{(1)}$ and $k_J^{(2)}$, it follows that (see eg. Kendall, Stuart & Ord 1994):

$$p^{(1)}(\delta) = \exp\left[\sum_{J=0}^{\infty} (-1)^J \frac{k_J^{(1)} - k_J^{(2)}}{J!} \frac{d^J}{d\delta^J} \right] p^{(2)}(\delta). \tag{6}$$

This equation is easy to prove by reobtaining the moments of $p^{(1)}$ through the generating function, having assumed that those of $p^{(2)}$ are given by $k_J^{(2)}$. After partial integration, the generating function of $p^{(1)}$ arises immediately, what proves the above equality (it is a nice exercise).

Equation (6) allows one to use the most convenient PDF to do the series expansion, $p^{(2)}(\delta)$. In particular, if one uses the Gaussian as the parent PDF one ends up with the Gram-Charlier series which yields the Edgeworth expansion in the perturbative limit (ie, when the variance of the PDF is small).

In this paper, we will focus on the Gamma distribution. In that case, we have that $p^{(2)}(\delta)$ is given by the Gamma PDF (see Eq(9) below) with $\delta = \beta z + \alpha$. Naming $p^{(i)}(\delta) = \bar{p^{(i)}}(z)$ $(i = 1, 2)$, we get

$$\frac{\bar{p^{(1)}}(z)}{\bar{p^{(2)}}(z)} = \exp\left[\sum_{J=0}^{\infty} (k_J^{(1)} - k_J^{(2)}) \left(\frac{-1}{\beta z} \right)^J L_J^{(p-J-1)}(z) \right] \tag{7}$$

However such a straightforward approach is ill-defined as the measure of the expansion is proportional to the order considered (the order of the expansion, J, is involved in the index denoting the order of the generalised Laguerre polinomials in Eq.[7]).

This will force us to modify the general method to derive a consistent expansion in terms of the relevant orthogonal polynomials (see §3).

2.2. *The Gaussian vs. the Gamma PDF*

In the case of the **Gaussian** (or normal $N(0, \sigma)$) PDF:

$$p(\delta) = p_G(\delta) \equiv \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{1}{2} \left(\frac{\delta}{\sigma} \right)^2 \right], \tag{8}$$

where σ is the rms standard deviation, the only parameter in this distribution. As the overdensity has to be positive $\rho > 0$, we have that $\delta < -1$, and a Gaussian PDF only makes physical sense when $\sigma \to 0$. For the Gaussian, $S_J = k_J = 0$ for $J > 2$.

The **Gamma** —also called negative binomial or Pearson Type III (PT3) PDF— arises from the Chi-Square distribution with N degrees of freedom when $1/\sigma^2 = N/2$ is taken to be a continuous parameter. This yields

$$p(\delta) = \phi(\delta) \equiv \frac{(1+\delta)^{\sigma^{-2}-1}}{\sigma^{2\sigma^2} \Gamma(\sigma^{-2})} \exp\left(-\frac{1+\delta}{\sigma^2} \right). \tag{9}$$

The hierarchical coefficients in Eq(4) are constant for all values of the variance in this case and give $S_J = (J-1)!$. These S_J values are equal to those of a simple exponential distribution. The Gamma PDF (or similar ones) has been found to be useful at describing the galaxy distribution (see eg, Fry 1986, Elizalde & Gaztañaga 1992, Gaztañaga & Yokohama 1993).

2.3. *The Saddle Point Approximation*

The moment generating function summarizes all the information concerning the higher-order cumulants, as long as the series expansion in terms of the latter converges. In the majority of the cases this is true (one counterexample to this rule is the Lognormal distribution). Thus, one may reconstruct the PDF from the the moment generating function in a consistent way as shown in Eq(5).

To obtain an asymptotic expansion of $p(\delta)$ for small δ, we introduce the Legendre transform,

$$\bar{\delta} \equiv d\psi(t)/dt, \quad G(\bar{\delta}) = \bar{\delta}t - \psi(t), \tag{10}$$

where the convexity of $G(\bar{\delta})$ is related to that of $\psi(t)$. Replacing this in the original expression for $p(\delta)$, we are left with

$$p(\delta) = \int_{G'=-i\infty}^{G'=+i\infty} \frac{G'' d\bar{\delta}}{2\pi} \exp[-\delta G'(\bar{\delta}) + \bar{\delta} G'(\bar{\delta}) - G(\bar{\delta})] \tag{11}$$

which is dominated by stationary points of the exponential at $\delta = \bar{\delta}$ (for real finite δ). The Saddle Point approximation of this integral is given in Fry (1985) and follows from the usual approach (Morse and Feshbach 1953):

$$p(\delta) \sim [G''(\delta)/2\pi]^{1/2} \exp[-G(\delta)]. \tag{12}$$

Once normalized, the distribution reads as

$$p(\delta) = \frac{[G''(\delta)/2\pi]^{1/2} \exp[-G(\delta)]}{\int_{-\infty}^{+\infty} [G''(\delta)/2\pi]^{1/2} \exp[-G(\delta)] d\delta}. \tag{13}$$

Provided one constructs the generating function $\psi(t)$ out of the irreducible moments,

$$\psi(t) = \sum_{n=2}^{\infty} \frac{\mu_n}{n!} t^n = \frac{1}{2}\sigma^2 t^2 + \frac{1}{6} S_3 \sigma^4 t^3 + \frac{1}{24} S_4 \sigma^6 t^4 + \cdots \tag{14}$$

(where the latter equality shows the expansion in terms of the hierarchical amplitudes), there is a general development for $G(\delta)$ in powers of δ which is derived by inverting the t variable in the Legendre transformation as to get $t = t(\delta)$, leading to

$$G(\delta) \approx \left[\frac{1}{2}\delta^2 - \frac{S_3}{6}\delta^3 + \frac{1}{8}\left({S_3}^2 - \frac{S_4}{3}\right)\delta^4 + \mathcal{O}(\delta^5)\right] \sigma^{-2}. \tag{15}$$

To get a proper expansion around the Gaussian PDF, we first need to arrange the exponential by factoring out the quadratic term. In the limit of small σ, we then get

$$\exp[-G(\nu)] \approx \left[1 + \frac{1}{3!} S_3 \nu^3 \sigma + \frac{1}{4!}\left(S_4 - 3{S_3}^2\right)\nu^4\sigma^2 + \frac{10}{6!}{S_3}^2 \nu^6 \sigma^2\right] p_G(\nu) + \mathcal{O}\left(\sigma^3\right), \tag{16}$$

where $\nu \equiv \delta/\sigma$. Applying the same expansion to the denominator (the normalization) and recalling the property for Gaussian integrals

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \nu^n e^{-\frac{\nu^2}{2}} d\nu = \begin{cases} (n-1)!! & \text{if } n \text{ even,} \\ 0 & \text{otherwise,} \end{cases} \tag{17}$$

we obtain a polynomial in powers of the small parameter σ^2 of the form:

$$\left[\int_{-\infty}^{+\infty} d\delta [G''(\delta)/2\pi]^{1/2} \exp[-G(\delta)]\right]^{-1} \simeq 1 - \left\{\frac{150}{6!}{S_3}^2 - \frac{3}{4!}S_4\right\}\sigma^2 + \mathcal{O}\left(\sigma^3\right). \tag{18}$$

Finally, we multiply both developments derived above and keep terms up to σ^3, to end up with

$$p(\nu) \simeq \left[1 + \frac{S_3}{3!} H_3(\nu)\sigma + \left\{\frac{1}{4!} S_4 H_4(\nu) + \frac{10}{6!}{S_3}^2 H_6(\nu)\right\}\sigma^2\right] p_G(\nu) + \mathcal{O}\left(\sigma^3\right), \tag{19}$$

$H_n(\nu)$ being the Hermite polynomials,

$$\begin{aligned}
H_3(\nu) &= \nu^3 - 3\nu, \\
H_4(\nu) &= \nu^4 - 6\nu^2 + 3, \\
H_5(\nu) &= \nu^5 - 10\nu^3 + 15\nu, \\
H_6(\nu) &= \nu^6 - 15\nu^4 + 45\nu^2 - 15, \\
H_7(\nu) &= \nu^7 - 21\nu^5 + 105\nu^3 - 105\nu, \\
H_9(\nu) &= \nu^9 - 36\nu^7 + 378\nu^5 - 1260\nu^3 + 945\nu, \ldots
\end{aligned}$$

which is the well-known (perturbative) Edgeworth series of a PDF up to third order. Higher-orders in the Edgeworth series can be obtained by keeping higher-orders in the Taylor expansions of Eqs(14),(15).

We stress that the latter expansion is derived under the assumption that the distribution is hierarchical, ie, the S_J are independent of σ. According to this, the Edgeworth expansion may be generalized to non-hierarchical PDFs whenever the scaling of $S_J(\sigma)$ is known and replaced in the generating function $\psi(t)$ from which the Saddle-Point approximation of $p(\delta)$ is built.

Non-linear PT for Gaussian initial conditions predicts corrective (σ dependent) terms to the leading order contribution to S_J of the form $S_J = S_J^{(0)} + S_J^{(1)}\sigma^2 + \mathcal{O}(\sigma^4)$, where $S_J^{(0)}, S_J^{(1)}$ are coefficients independent of σ. This must be taken into account to make consistent predictions from non-linear dynamics for the third-order term (or higher) in the Edgeworth series of the PDF (see Bernardeau & Kofman 1995).

The Edgeworth expansion —or any other expansion based on the symmetry around the peak of the PDF being approximated— is only a good candidate for fitting the evolutionary picture of the density profile as a first order approach, because high-density (non-Gaussian) exponential tails develop in further stages of the non-linear evolution for arbitrary initial conditions.

The Edgeworth series has also been applied to non-linear transformations of the Gaussian process to fit the exponential tails observed in the simulations as the system evolves. The 'skewed' Lognormal approximation put forward by Colombi (1994) is an example of this scheme which takes advantage of the apparently privileged role the Lognormal PDF plays among the non-Gaussian ones. This is suggested by the integration of the continuity equation in Lagrangian coordinates on one hand, and by the good fit to the observational PDF on intermediate scales (related to a $n = 1$ spectral index for a power-law power spectrum), on the other hand.

Nevertheless, it is still lacking in the literature a well-defined expansion around a non-Gaussian PDF which may be better suited than the Gaussian to model the gravitational evolution of cosmic fluctuations in the weakly non-linear regime. It is not clear yet whether the initial conditions of structure formation in the universe were Gaussian (as suggested by standard inflationary models), or not. In the latter case, it is necessary to investigate expansions around non-Gaussian PDFs if one wants to describe clustering. These issues have been our main motivation to introduce an alternative expansion around a non-Gaussian PDF.

In the next section we derive a general expansion around the Gamma distribution, ie., around an arbitrary exponential tail, making use of the completeness and orthogonality properties of the Laguerrre polynomials. They are formally analogous to the Hermite polynomials that appear in the Edgeworth series around a Gaussian.

3. EXPANSION AROUND THE GAMMA PDF

Our starting point here will be the expansion of the PDF in terms of the Gamma distribution, with a basis that will be given by generalized Laguerre polynomials (instead of the Hermite ones in Eq(20)). That such an expansion makes sense becomes clear from the fact that the Gamma distribution is proportional to the measure associated with this particular family of orthogonal polynomials. In this sense, the Gamma expansion is formally reminiscent of the Edgeworth series as the main difference consists of replacing the Hermite polynomials by the Laguerre ones as the basis for the expansion. Notice however that while the Edgeworth series might be built from the Gram-Charlier series (Cramér 1946), given by the Gaussian PDF and its derivatives (see Juszkiewicz et al. 1995), here we cannot simply take successive derivatives of the Gamma distribution. This is because, in doing so, we would actually fail to make contact with a consistent theory of orthogonal polynomials. In short, if the generalized Laguerre ones are to be used (and those are the only possibility in the case considered), it turns out that the integration measure changes with the *order* of the generalized polynomial (and not just with the parameter of the family), and this would invalidate the whole approach. We should, by the way, recall the good properties of a well defined expansion in terms of an orthogonal basis of polynomials, that are orthonormalized with respect to an scalar product defined by a fixed integration measure. This is a rigorously defined mathematical theory, that in the case of the Gaussian measure happens to coincide with the Taylor expansion in terms of the derivatives of the function. In the case of the Gamma function, on the contrary, it turns out that the two expansions do not coincide, and only the one in terms of orthogonal polynomials has rigorous mathematical sense. Thus, this must be the starting point in our approach, as the Taylor expansion (given in terms of the successive derivatives) must be relegated to a mere formal expansion lacking adequate convergence properties.

The key point in our scheme is to build a general and consistent expansion in terms of the relevant orthogonal polynomials (the Laguerre ones, in the present case), *not necessarily* given by derivatives of the parent PDF (ie, we do not make use of Equation (6)). Bearing this in mind, we modify the approach based on the general Eq(6) and define an expansion of the PDF in terms of the Gamma distribution as follows:

$$p(\mu) \equiv \sum_{n=0}^{\infty} c_n L_n^{(p-1)}(\mu)\phi(\mu), \tag{20}$$

ie, we define an expansion for which all orthogonal polinomials are well defined, with coefficients:

$$c_n = \frac{n!\Gamma(p)}{\Gamma(n+p)} \int_0^{\infty} p(\mu) L_n^{(p-1)}(\mu)\, d\mu, \tag{21}$$

being $\phi(\mu)$ the Gamma PDF:

$$\phi(\mu) d\mu = \frac{1}{\Gamma(p)} \mu^{p-1} e^{-\mu} d\mu, \tag{22}$$

$$\mu = \frac{x-\alpha}{\beta} \geq 0. \tag{23}$$

This is actually a three-parameter (p, α and β) family of distributions out of which only one, p, is relevant for normalized variables (such as density fluctuations, δ).

The generalized Laguerre polynomials we shall need are given by:

$$L_n^{(p-1)}(\mu) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \binom{n+p-1}{n-k} \mu^k, \tag{24}$$

in particular,

$$\begin{aligned}
L_1^{(p-1)}(\mu) &= p - \mu, \\
L_2^{(p-1)}(\mu) &= \frac{p(p+1)}{2} - (p+1)\mu + \frac{\mu^2}{2}, \\
L_3^{(p-1)}(\mu) &= \frac{p(p+1)(p+2)}{6} - \frac{(p+1)(p+2)}{2}\mu \\
&\quad + \frac{p+2}{2}\mu^2 - \frac{\mu^3}{6},
\end{aligned} \tag{25}$$

and the coefficients c_n are easy to calculate:

$$\begin{aligned}
& c_0 = 1, \quad c_1 = c_2 = 0, \\
& c_3 = -\frac{\Gamma(p+1)}{\Gamma(p+3)}(S_3 - 2!), \\
& c_4 = \frac{\Gamma(p+1)}{\Gamma(p+4)}\left[(S_4 - 3!) - 12(S_3 - 2!)\right], \\
& c_5 = \frac{\Gamma(p+1)}{\Gamma(p+5)}\left[-(S_5 - 4!) + 20(S_4 - 3!) - 120(S_3 - 2!)\right].
\end{aligned} \tag{26}$$

In general, one finds:

$$\begin{aligned}
c_n = \frac{\Gamma(p+1)}{\Gamma(p+n)} \sum_{k=3}^{n} \Big\{ & (-1)^k a_{n,k} \left[S_k - (k-1)!\right] \\
& + b_{n,k} \left[S_k - (k-1)!\right]^2 + \cdots \Big\}.
\end{aligned} \tag{27}$$

It is clear that for $S_k = (k-1)!$ we recover the Gamma PDF, $p(\mu) = \phi(\mu)$.

Note that for the first coefficients it holds

$$\begin{aligned}
a_{n,n} &= 1, \\
a_{n,n-1} &= n(n-1), \ldots
\end{aligned}$$

We should stress the fact that for higher orders (ie., for c_6 or higher) there appear linear terms in $p = 1/\sigma^2$ which couple to those quadratic in $[S_k - (k-1)!]$. For instance,

$$c_6 = \frac{\Gamma(p+1)}{\Gamma(p+6)}\left[(S_6 - 5!) - 30(S_5 - 4!) + 300(S_4 - 3!)\right.$$
$$\left. -\ 1200(S_3 - 2!) + 10\,p\,(S_3 - 2!)^2\right]. \tag{28}$$

Notice that an appropriate expansion for small σ will have contributions from these higher-order terms. In order to see clearly which is the parameter in the expansion (20), and set up a comparison with the Edgeworth expansion, we express Eq(20) in terms of the same variable, $\nu = \delta/\sigma$, so that $\mu = 1/\sigma^2 + \nu/\sigma$. Doing this, we have:

$$p(\nu) = \left\{1 + \sum_{n=3}^{\infty} \frac{\Gamma(1+1/\sigma^2)}{\Gamma(n+1/\sigma^2)} L_n^{(1/\sigma^2-1)}(\mu) \times \sum_{k=3}^{n} (-1)^k a_{n,k}[S_k - (k-1)!]\right\}\phi(\nu)$$
$$= \left\{1 + \sum_{n=3}^{\infty} \sigma^{n-2}\, F_n \times \sum_{k=3}^{n}(-1)^{n-k} a_{n,k}[S_k - (k-1)!]\right\}\phi(\nu), \tag{29}$$

with the $F_n = F_n(\nu, \sigma)$ being of the form:

$$F_n(\nu,\sigma) = \frac{1}{n!}H_n(\nu) + \mathcal{Q}_n(\nu)\,\sigma + \mathcal{R}_n(\nu)\,\sigma^2 + \mathcal{O}(\sigma^3) \tag{30}$$

with H_n the Hermite polynomial of order n, and

$$\mathcal{Q}_3(\nu) = \frac{2}{3} - \nu^2$$
$$\mathcal{Q}_4(\nu) = \frac{\nu}{6}\left(7 - 3\nu^2\right)$$
$$\mathcal{Q}_5(\nu) = \frac{1}{6}\left(-2 + 5\nu^2 - \nu^4\right)$$
$$\mathcal{Q}_6(\nu) = \frac{\nu}{72}\left(-33 + 26\nu^2 - 3\nu^4\right)$$
$$\mathcal{R}_3(\nu) = \frac{\nu}{2}\left(5 - \nu^2\right), \ldots \tag{31}$$

To summarize, we see that what we have in Eq(29) is in fact an expansion in terms of the Gamma PDF in the perturbative limit (ie., when σ is small), in formal analogy to the Edgeworth expansion (which uses the Gaussian as the parent PDF). The new expansion should presumably be much better suited than the Edgeworth expansion, to parametrize PDFs with exponential tails.

4. COMPARISON OF THE EDGEWORTH & GAMMA EXPANSIONS

We can now compare the Gamma with the Edgeworth series. We just have to replace Eqs(30),(31) in Eq(29), and express the (third order) Gamma expansion as a series in σ, the rms standard deviation. We can write it in the following compact notation:

$$\frac{p(\nu)}{\phi(\nu)} = 1\ +\ H_3(\nu)\Delta_3\ \sigma$$
$$+\ \left\{H_4(\nu)(\Delta_4 - 3\Delta_3)\ + H_6(\nu)\Delta_3^2/2 + 6\,\mathcal{Q}_3(\nu)\Delta_3\right\}\sigma^2$$
$$+\ \left\{H_5(\nu)(\Delta_5 - 4\Delta_4 + 6\Delta_3) + H_7(\nu)(\Delta_4\Delta_3 - 3\Delta_3^2)\right.$$
$$+\ H_9(\nu)\Delta_3^3/6 + 24\mathcal{Q}_4(\nu)(-3\Delta_3 + \Delta_4)$$
$$\left.+\ 360\mathcal{Q}_6(\nu)\Delta_3 + 6\mathcal{R}_3(\nu)\Delta_3\right\}\sigma^3 + \mathcal{O}(\sigma^4) \tag{32}$$

where $\phi(\nu)$ is the Gamma distribution (see Eq(9)) for the dimensionless variable ν. The polynomials $\mathcal{Q}_n, \mathcal{R}_n$ are given above, Eq(31), and the reduced moments of the $p(\nu)$ PDF, S_J, appear as differences with the moments, $S_J^{(p)}$ of the parent PDF (over which we are expanding):

$$\Delta_J \equiv \frac{S_J - S_J^{(p)}}{J!}. \tag{33}$$

In particular, for the Gamma expansion, $S_J^{(p)} = (J-1)!$. Eq.[gammaexp] is the main result of this paper.

For the Edgeworth series around the Gaussian one has up to the same order,

$$\frac{p(\nu)}{p_G(\nu)} = 1\ +\ H_3(\nu)\Delta_3\ \sigma$$
$$+\ \{H_4(\nu)\Delta_4\ + H_6(\nu)\Delta_3^2/2\}\sigma^2 \tag{34}$$
$$+\ \{H_5(\nu)\Delta_5 + H_7(\nu)\Delta_4\Delta_3 + H_9(\nu)\Delta_3^3/6\}\sigma^3,$$

where $p_G(\nu)$ is the Gaussian PDF with $S_J^{(p)} = 0$.

By comparing the Gamma and the Edgeworth expansions given above, we see that the Gamma expansion recovers all the terms that appear in the Edgeworth expansion plus some corrective terms. In general the Gamma expansion has, by construction, exponential tails and a better general behaviour than the Edgeworth expansion, both on the positivity of $p(\nu)$ and the variate itself, ρ.

Notice that, as suggested by expansions Eqs(32),(35), the convergence of these series depends on the magnitude of the coefficients $\Delta_J\,\sigma^{J-2}$ that weight the contribution from every polinomial. This is also true for the Edgeworth expansion, where the natural smallnes parameter is given by $S_J\,\sigma^{J-2}$ (see also Juszkiewicz et al. (1995)). This can already be guessed from the Saddle Point approach to the Edgeworth series. In particular (see Eq(14)), the cumulant generating function, $\psi(t)$, is expanded in terms of powers of σt, with coefficients given by the cumulants $k_J \equiv S_J\,\sigma^{J-2}$. Accordingly, in the Gamma expansion, the convergence depends on how close the tail of the PDF is to an exponential one within the perturbative limit.

Figure 1 (left panel) shows a comparison of the two expansions to leading order in σ, for $\sigma = 0.2 - 0.3$ and $S_3 = 3 - 4$, as labeled in the Figures. For reference we also show the Gaussian distribution with the same σ (dotted line). The Edgeworth expansion (continuous lines) quickly develops negative probabilities (which are shown in absolute value) for negative values of $\nu \equiv \delta/\sigma$. It is also clear how the Edgeworth has Gaussian tails dropping quickly to zero, while the Gamma expansion (shown as a dashed line) has exponential-type tails.

Figure 1 also shows a comparison of the two expansions to the next order (second order) in σ, for $\sigma = 0.2 - 0.3$, $S_3 = 4$, and $S_4 = 15 - 20$ as labeled in the Figures. The Edgeworth expansion (continuous lines) shows now more negative values of the probability (shown in absolute

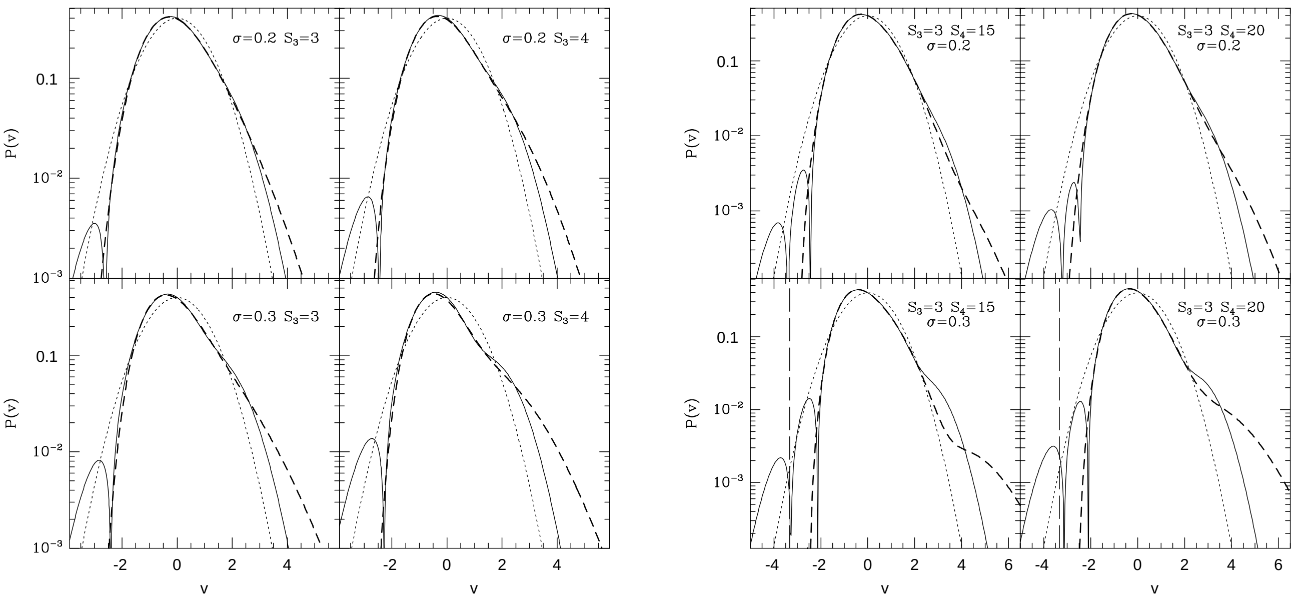

FIG. 1.— Comparison of the leading order (left panel) and second order (right panel) Edgeworth and Gamma PDF expansions as functions of $\nu \equiv \delta/\sigma$ for several values of σ, and S_J as labeled in the Figures. The dotted, dashed and continuous lines correspond to the Gaussian PDF, Gamma and Edgeworth expansions, respectively.

value) for negative values of $\nu \equiv \delta/\sigma$, and large modulations for $\nu > 0$. For some range of parameters and $\nu > 0$ (ie. $\delta > 0$), the Gamma PDF also exhibits negative probabilities but they are typically much smaller than in the Edgeworth expansion. This problem was less apparent in Fig. 1, when working at the first order of the series (left panel).

The long-dashed vertical line corresponds to $\nu = -1/\sigma$, which marks the range of positive density: $\rho > 0$ or $\nu > -1/\sigma$. This line is crossed both by the Gaussian PDF and the Edgeworth PDF, indicating that these distributions cannot be used as physical distributions above some small value of σ.

5. COMPARISON WITH SIMULATIONS

As it is suggested by Eqs(32),(35), differences in both expansions might be slight, specially around the peak of the PDF, as to first order both expansions are formally equivalent. Thus, which one best fits numerical results is a matter of careful analysis. The Edgeworth expansion has been shown to provide a very good approximation to model the PDF resulting from the weakly non-linear gravitational growth from small Gaussian initial fluctuations. We next compare both expansions with N-body simulations to see if we find significant deviations in their predictions.

We measure the PDF in 10 realizations of SCDM simulations, $\Omega = 1$ and $\Gamma = 0.5$, with $L = 180\,h^{-1}$ Mpc and $N = 64^3$ particles and $\sigma_8 = 1$ (Croft & Efstathiou 1994). We consider several smoothing radius which correspond to different values of the variance, σ^2. The errors correspond the rms standard deviation in the 10 realizations.

Figure 2 corresponds to $\sigma \simeq 0.2$. As can be seen in the plot both expansions including the second order produce very similar results, within the error bars. Note also that they both are significantly different from the Gaussian result (dotted line). Similar results are obtained for lower values of σ.

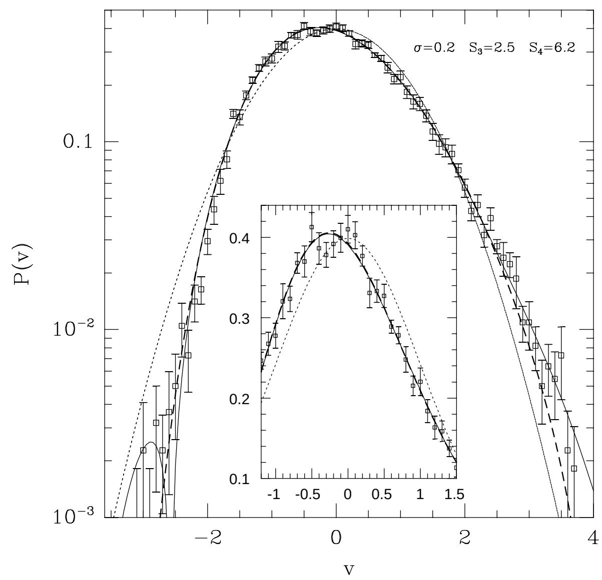

FIG. 2.— Comparison of gravitational simulations with the second order Edgeworth and Gamma PDF expansions as a function of $\nu \equiv \delta/\sigma$. We use as parameters the measured values of σ, S_3 and S_4 as labeled in the Figure. The dotted, dashed and continuous lines correspond to the Gaussian, Gamma and Edgeworth distribution expansions. The inset shows a detail around the peak in linear scale.

When σ is small, $\sigma \lesssim 0.3$, using the measured N-body (non-linear) values of σ and S_J as parameters for the PDF expansion, it yields very similar numbers, within the error bars, to the ones obtained using the corresponding linear σ and non-linear PT predictions for S_J. The latter is

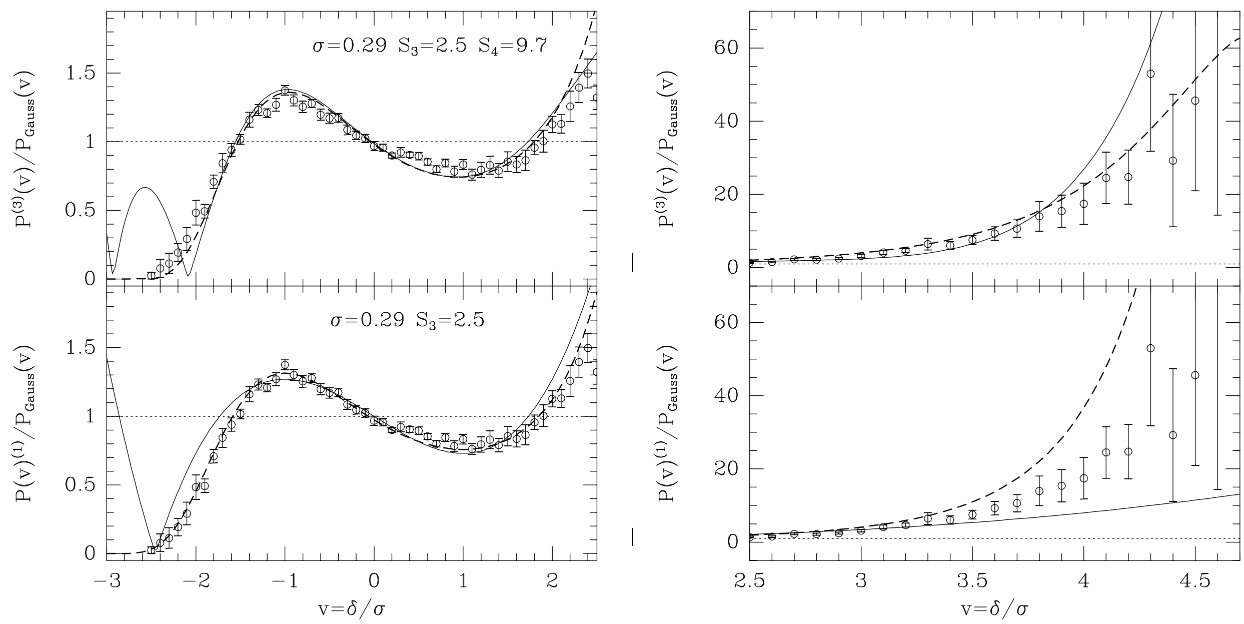

FIG. 3.— Deviations from the Gaussian PDF for both expansions and in N-body simulations (symbols). The lower panels displays results for the first order. The upper panels show the expansions including the third-order terms. The solid line is given by the Edgeworth series while the dashed one shows the Gamma expansion. The left and right panels show different ranges in ν.

in agreement with what was suggested by Juszkiewicz et al. (1995), Bernardeau & Koffman (1995). For larger values of the *rms* fluctuation, $\sigma \gtrsim 1$, differences in the PDF when using perturbative or non-linear parameters become larger.

In Figure 3 we can see that for $\sigma \simeq 0.3$ the Gamma expansion seems to be in better agreement with the numerical results than the Edgeworth specially for negative values of ν. It is seen that both expansions yield similar results, except for the negative tail which is better recovered by the Gamma expansion again. The Gamma distribution seems to perform slightly better also around $\nu \simeq 1-5$ as can be seen in Figure 3 (right panel). For small ν the Edgeworth series yields relatively large negative probabilities of the order $P(\nu) \simeq 0.01$. Thus, in this case it would be better to use the Gamma expansion if our priority is a better overall behaviour. This needs not be rigorously generic and the situation could change if we explore a different domain of the parameter space. It mainly depends on how large the values of Δ_J are for the case of interest. Also note that larger values of σ require higher orders in the expansions, which could then change their relative performances.

Figure 4 illustrates what happens for larger values of σ, where these expansions are expected to break down. Only the first order is shown, but the agreement does not improve for higher orders. In this regime, $\sigma \simeq 1$, the expansions do not work well, as expected, and one has to use some different approach to model the PDF. The short-dashed line shows the results of one such model: the non-linear Spherical Collapse model (from Fig.7 in Gaztanaga & Croft 1999). Errobars in the simulations are comparable to the size of the symbols. The Spherical Collapse model for the evolution of density perturbations can be used as a local Lagrangian mapping to relate the initial and evolved fluctuation. The only input required is the linear variance σ and its slope γ (eg see Fosalba & Gaztañaga 1998a,b, Gaztañaga & Fosalba 1998). Notice however that, while the Edgeworth and Gamma expansions could be used to model an arbitrary PDF, given its moments S_J, the Spherical Collapse is only intended to model gravitational dynamics of cosmic fluctuations.

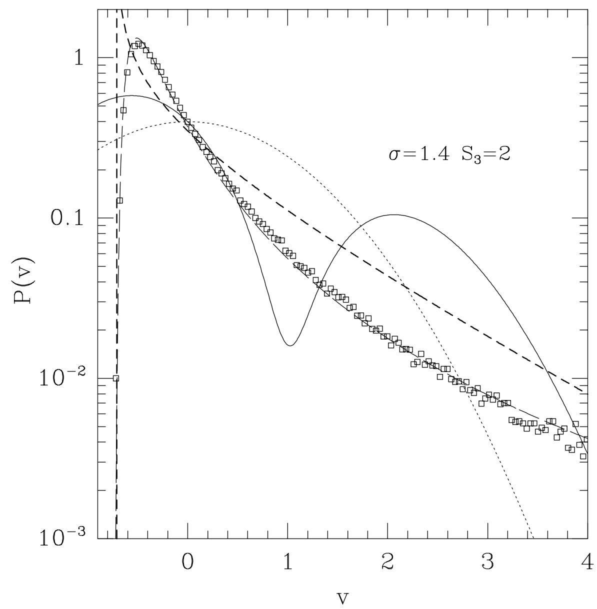

FIG. 4.— Comparison of Nbody results (symbols) with the Gaussian (dotted line), the Edgeworth series (continuous) and the Gamma expansion (short dashed line). Both expansions are given to first order. The long-dashed line following closely the simulation points corresponds to the spherical collapse model.

6. DISCUSSION & CONCLUSION

In this paper we have introduced the Gamma expansion, as an alternative to the well-known Edgeworth series, to model the gravitational evolution of the density field on large-scales in terms of its lower order moments $S_J \equiv k_J/k_2^{J-1}$ (see Eq(4)). The search for an alternative expansion is motivated by the fact that the Gaussian PDF, which is used as the parent distribution for the Edgeworth series, is not strictly well-defined for describing positive variates, such as the density field. As a consequence, the expansion built out of the Gaussian exhibits undesirable properties, namely, it predicts negative probabilities and allows for negative densities. In §2 we have provided an independent derivation of the Edgeworth series based on the saddle point approximation to the Legendre transform of the PDF.

In the case of the Gamma expansion (see §3), the basis for the series is given by the generalised Laguerre polinomials. These are the counterparts to the Hermite polinomials which appear for expansions around the Gaussian, and allow for the construction of a consistent series in the perturbative limit (when the variance is small). The coefficients in both expansions can be writen in terms of $\Delta_J \equiv (S_J - S_J^{(p)})/J!$, where the reduced cumulants of the PDF, S_J, appear as differences with the corresponding cumulans, $S_J^{(p)}$ of the parent PDF (over which we are expanding), ie. $S_J^{(p)} = 0$ for a Gaussian PDF, or $S_J^{(p)} = (J-1)!$ for the Gamma PDF (see Eq(32) and Eq(35)). The Gamma expansion recovers all the contributions that appeared in the Edgeworth series, as functions of Δ_J, plus some corrective terms, for second order in σ or higher. The convergence of the series is safe as long as the PDF develops a tail close enough to exponential, ie. when Δ_J are small. Gravitational clustering predicts the appearence of such exponential tails in the weakly non-linear evolution of the cosmic density field, at least for Gaussian initial conditions. Therefore, the expansion introduced in this paper should constitute a good candidate to properly model clustering on large scales.

We have carried out a detailed comparison of the performance of the Edgeworth and the Gamma expansions in the perturbative regime with respect to cosmological N-body simulations with $\sigma \lesssim 0.4$. We have found that they both yield a very similar agreement with numerical results around the peak of the distribution. The Gamma expansion provides a better general match to the PDF on the tails. In particular, the negative density tail measured in N-body simulations is accurately recovered from the Gamma expansion, unlike the case of the Edgeworth series. Nevertheless, in general, the performance of the expansions depend strongly on the values of Δ_J. The agreement is better for the expansion which has its parent moments $S_J^{(p)}$ closer to those of the PDF we want to model. In other words, the smaller the Δ_J, the better the behaviour of the expansion of the PDF.

In summary, the Gamma expansion provides an interesting alternative to the Edgeworth series without introducing any additional mathematical entanglement. Both expansions have the same inputs (the cumulants of the PDF we want to model) and outputs (the recovery of the full PDF) and similar expressions (see Eq(32) and Eq(35)). The proposed Gamma expansion is better suited for describing a real PDF, because it always yields positive densities and the PDF is effectively positive-definite. It is also possible to expent the Gamma expansion to a multivariate form, using a generalization of the Laguerre polinomials. Finally, we stress that many of the arguments presented here are rather generic and they might be useful when adressing the problem of modeling PDFs in other contexts.

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